INDIAN INSTITUTE OF TECHNOLOGY DELHI

MAL 111: Introduction to Analysis and Differential Equations

Minor I 2007-08 (I Semester)

Maximum Marks: 25 Time: 1 Hour

Give complete statements of the results used.

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1. Consider the following two metrics on $\mathbb{R}^2$, where for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, d_1 and d_2 are defined by

$$d_1\big((x_1,x_2),(y_1,y_2)\big) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$
$$d_2\big((x_1,x_2),(y_1,y_2)\big) = |x_1 - y_1| + |x_2 - y_2|.$$

Show that every open sphere of the metric space $(\mathbb{R}^2, d_1)$ contains an open sphere of $(\mathbb{R}^2, d_2)$ and, conversely, every open sphere of the metric space $(\mathbb{R}^2, d_2)$ contains an open sphere of $(\mathbb{R}^2, d_1)$.

[5]

2. (a) Let $\{x_n\}$ and $\{y_n\}$ be sequences in the metric space (X, d) such that $x_n \to x$ and $y_n \to y$ in X and let a be a real number. If $d(x_n, y_n) < a$ for each $n = 1, 2, 3 \ldots$, show that $d(x, y) \leq a$.

   (b) Let (X, d) be a complete metric space and Y be a closed subspace of X. Show that Y is complete.

[3+3]

3. (a) Let X and Y be metric spaces and A a non-empty subset of X. Let $f : X \to Y$ and $g : X \to Y$ be continuous mappings such that $f(x) = g(x)$ for every $x \in A$. Show that $f(x) = g(x)$ for every $x \in \overline{A}$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that there exists a real number α such that $f(x) = \alpha x$ for every $x \in \mathbb{R}$.

   Recall that a subset A of a metric space X is said to be dense in X if $\overline{A} = X$. You can use the fact that the set of rational numbers is dense in $\mathbb{R}$.

[3+4]

4. (a) Show that every sequentially compact metric space is compact.

   (b) Let A be a bounded subset of $\mathbb{R}^n$. Show that $\overline{A}$ is compact.

[4+3]
